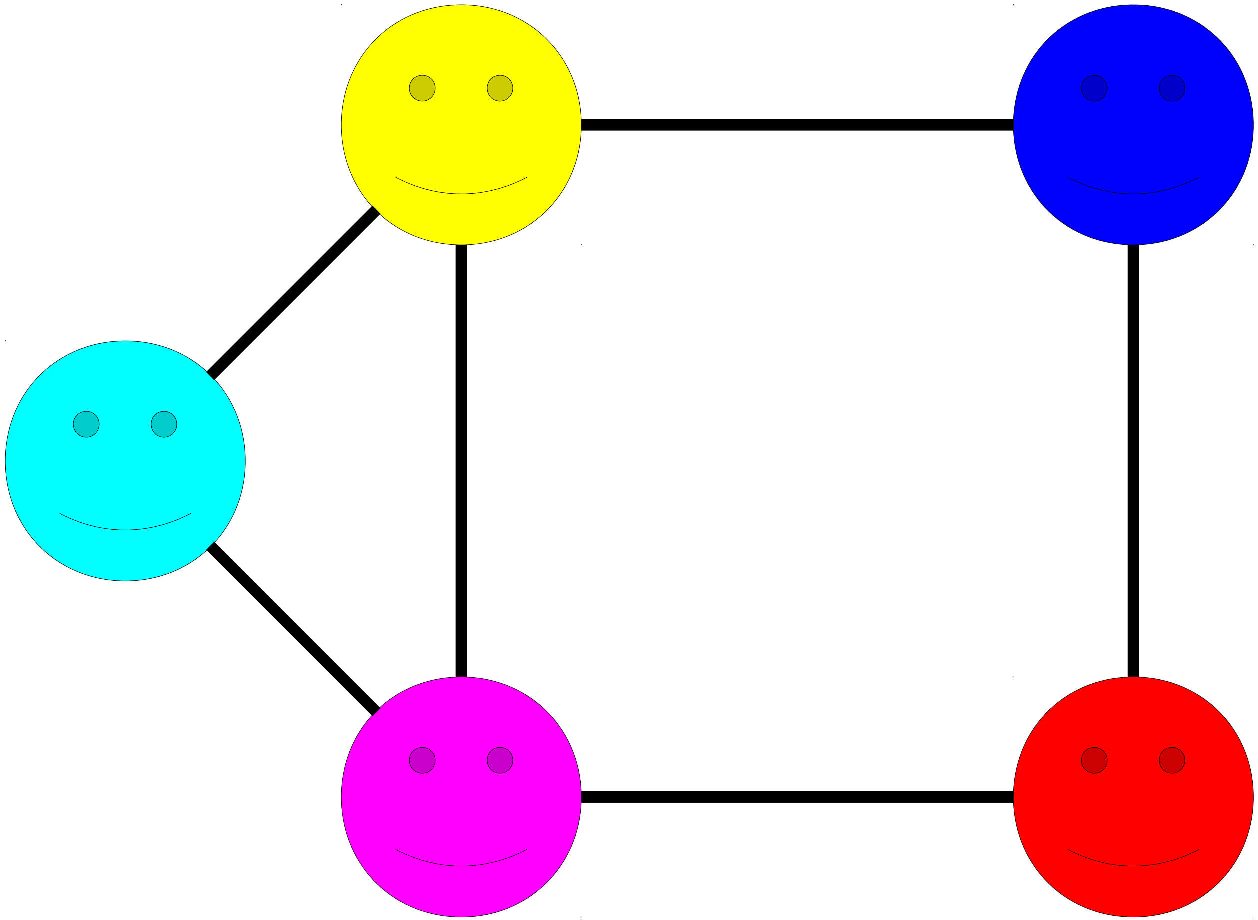
Graph Theory

Part Two

Recap from Last Time

A ***graph*** is a mathematical structure for representing relationships.



A graph consists of a set of ***nodes*** (or ***vertices***) connected by ***edges*** (or ***arcs***)

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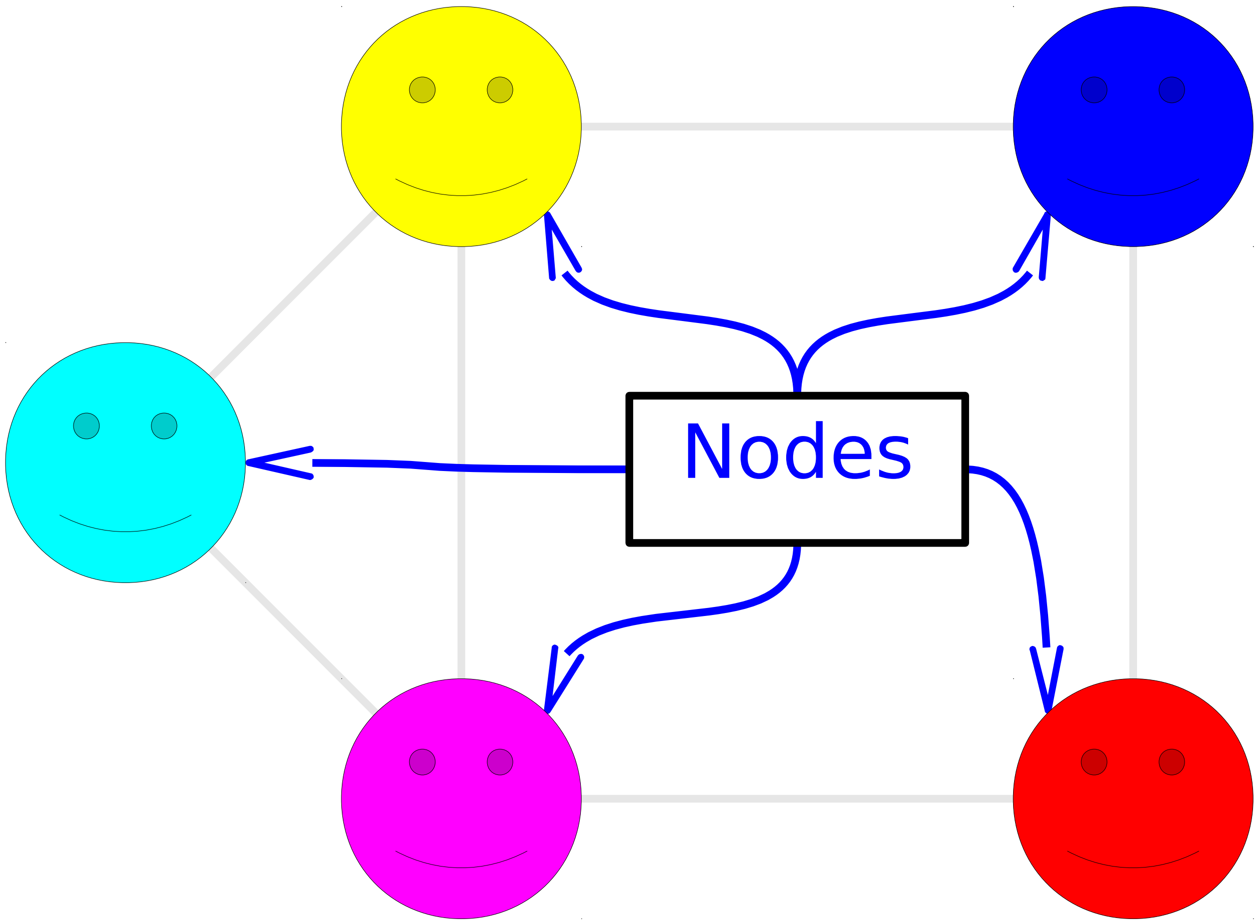

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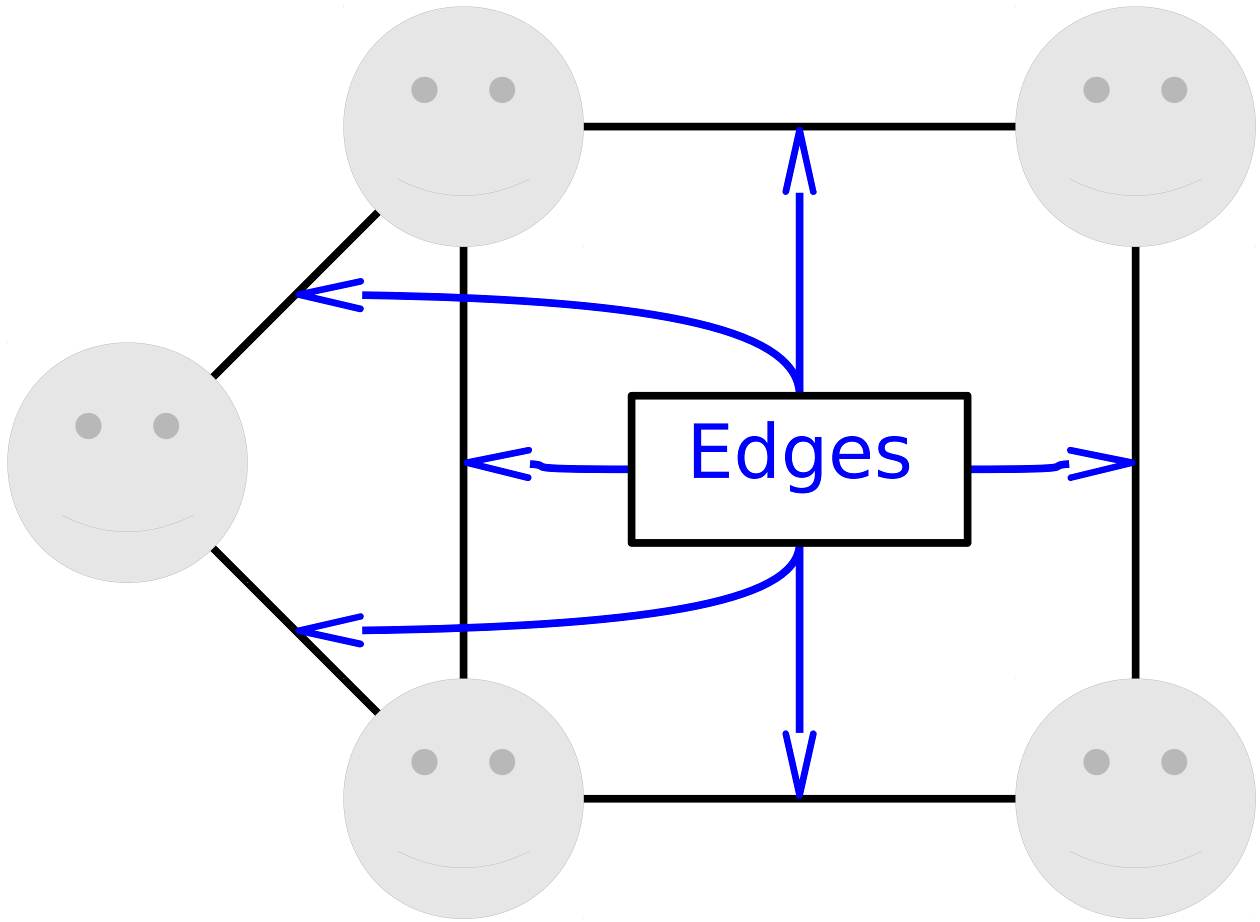

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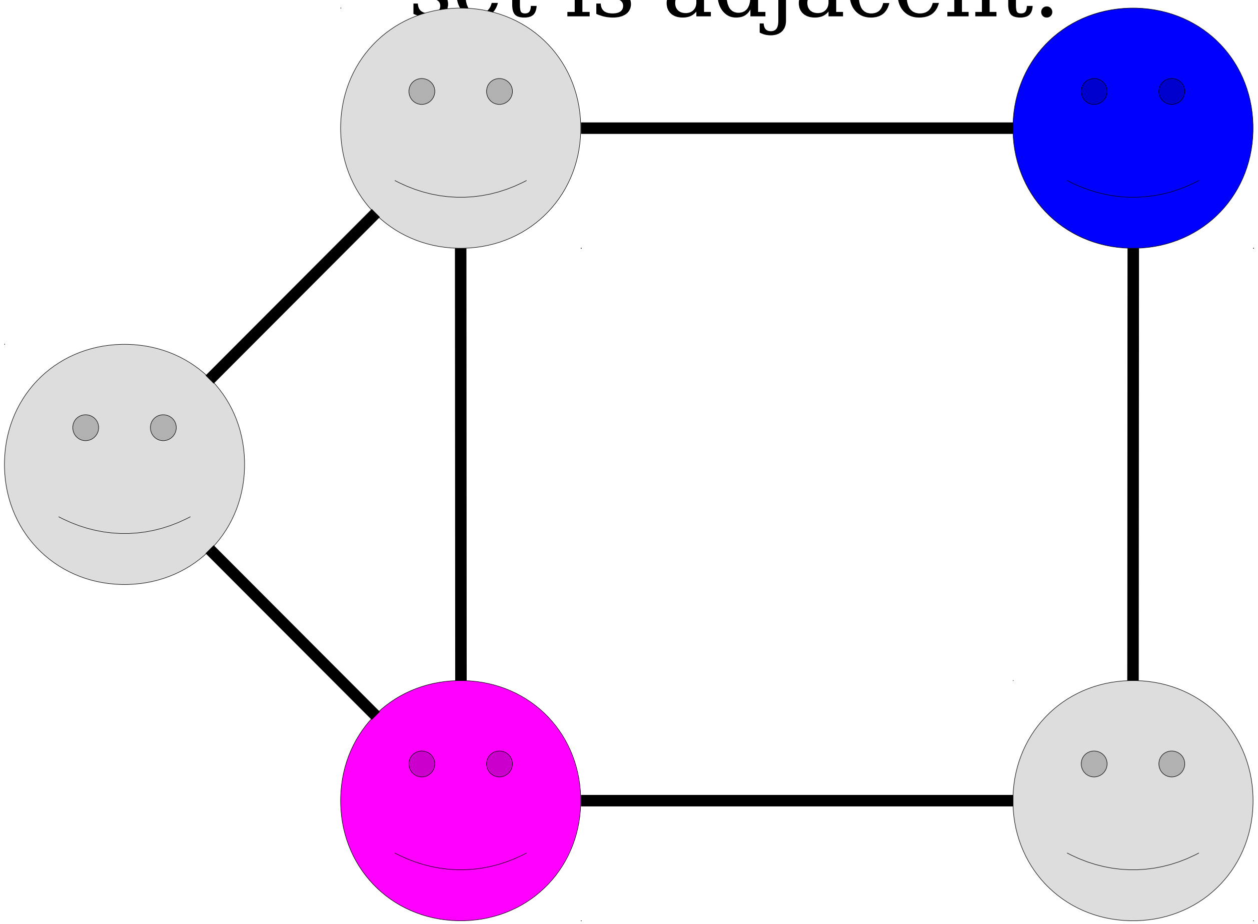
Adjacency and Connectivity

- Two nodes in a graph are called ***adjacent*** if there's an edge between them.
- Two nodes in a graph are called ***connected*** if there's a path between them.
  - A path is a series of one or more nodes where consecutive nodes are adjacent.
- An entire graph is called ***connected*** if every pair of nodes in the graph is connected.

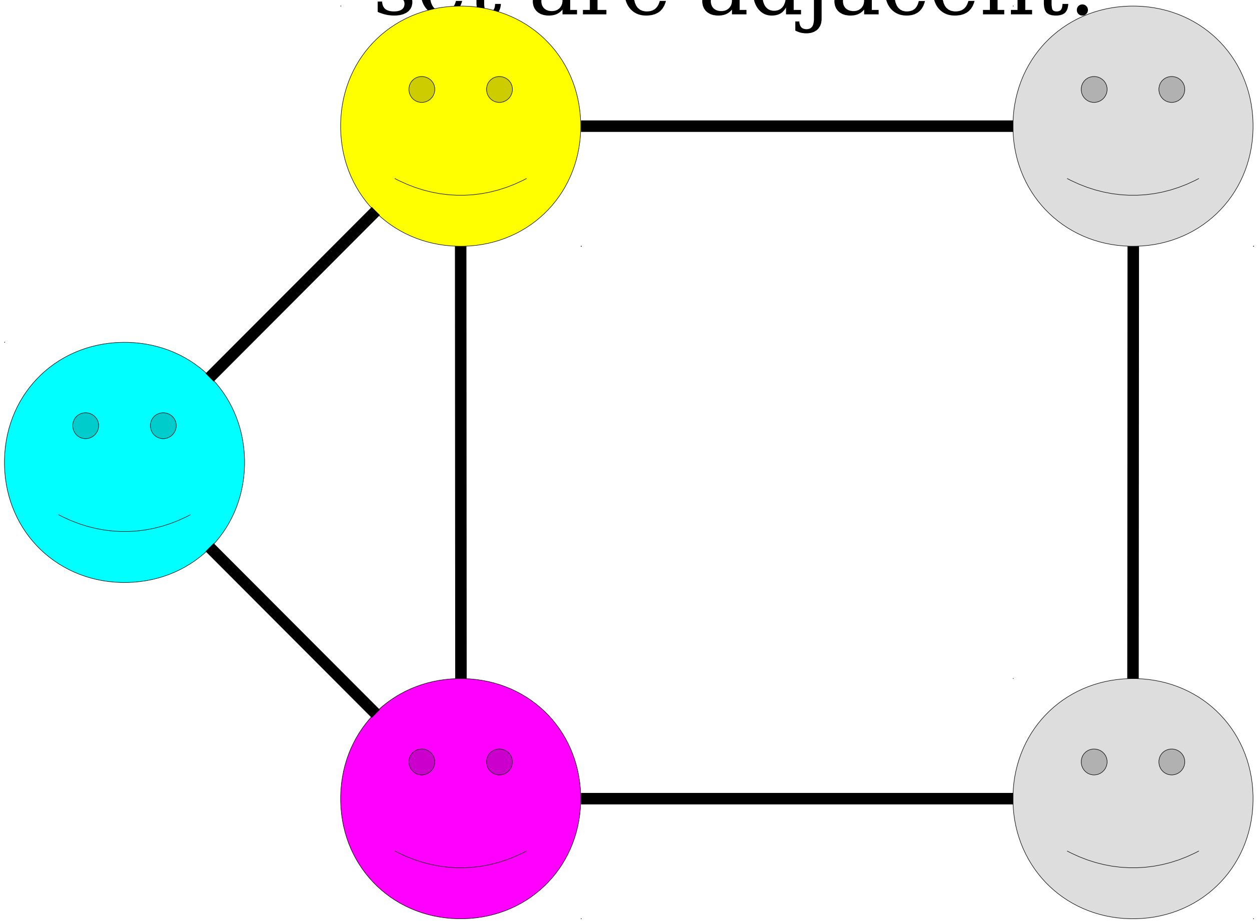
New Stuff

An ***independent set*** is a subset of nodes of a graph such that no pair of nodes in the set is adjacent.



Here is one independent set of size two.

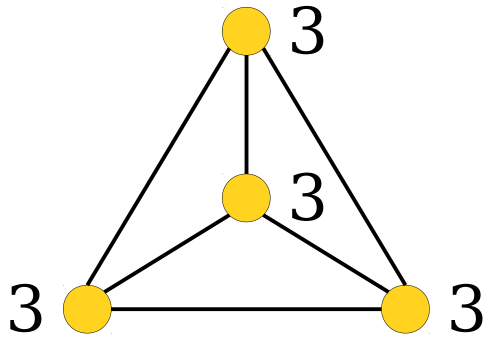
An ***k-clique*** is a subset of nodes of a graph such that all pairs of distinct nodes in the set are adjacent.



Here is a 3-clique.

Degrees

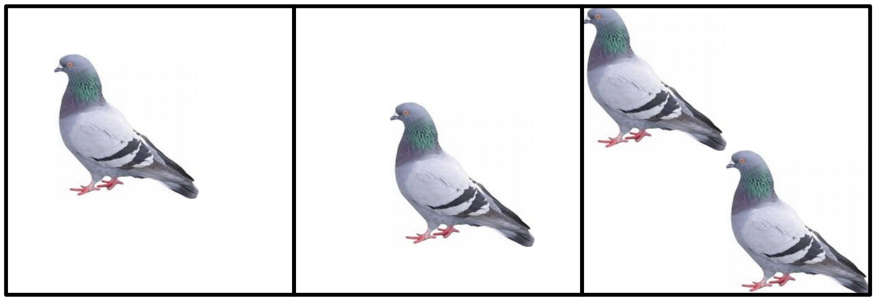
- The ***degree*** of a node *v* in a graph is the number of nodes that *v* is adjacent to.



The Pigeonhole Principle

The Pigeonhole Principle

- ***Theorem (The Pigeonhole Principle):*** If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.



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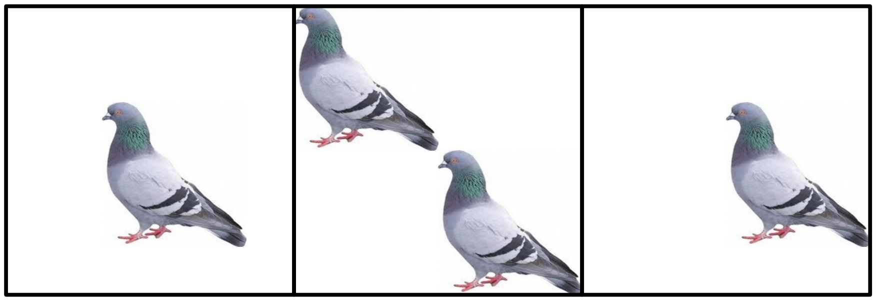

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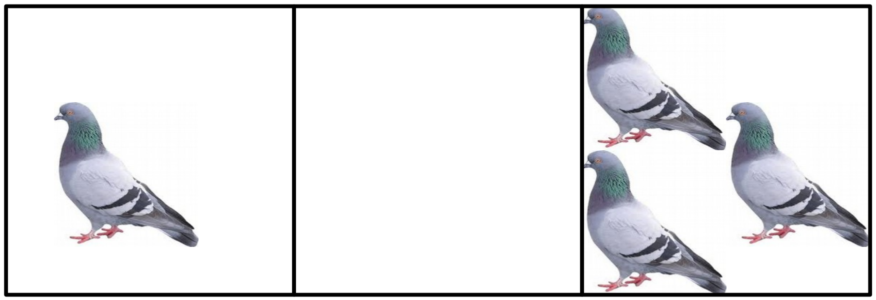

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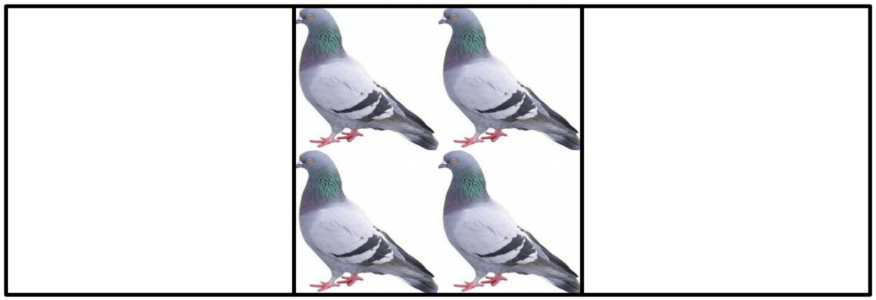


Some Simple Applications

- Any group of 367 people must have a pair of people that share a birthday.
  - 366 possible birthdays (pigeonholes)
  - 367 people (pigeons)
- Two people in San Francisco have the exact same number of hairs on their head.
  - Maximum number of hairs ever found on a human head is no greater than 500,000.
  - There are over 800,000 people in San Francisco.

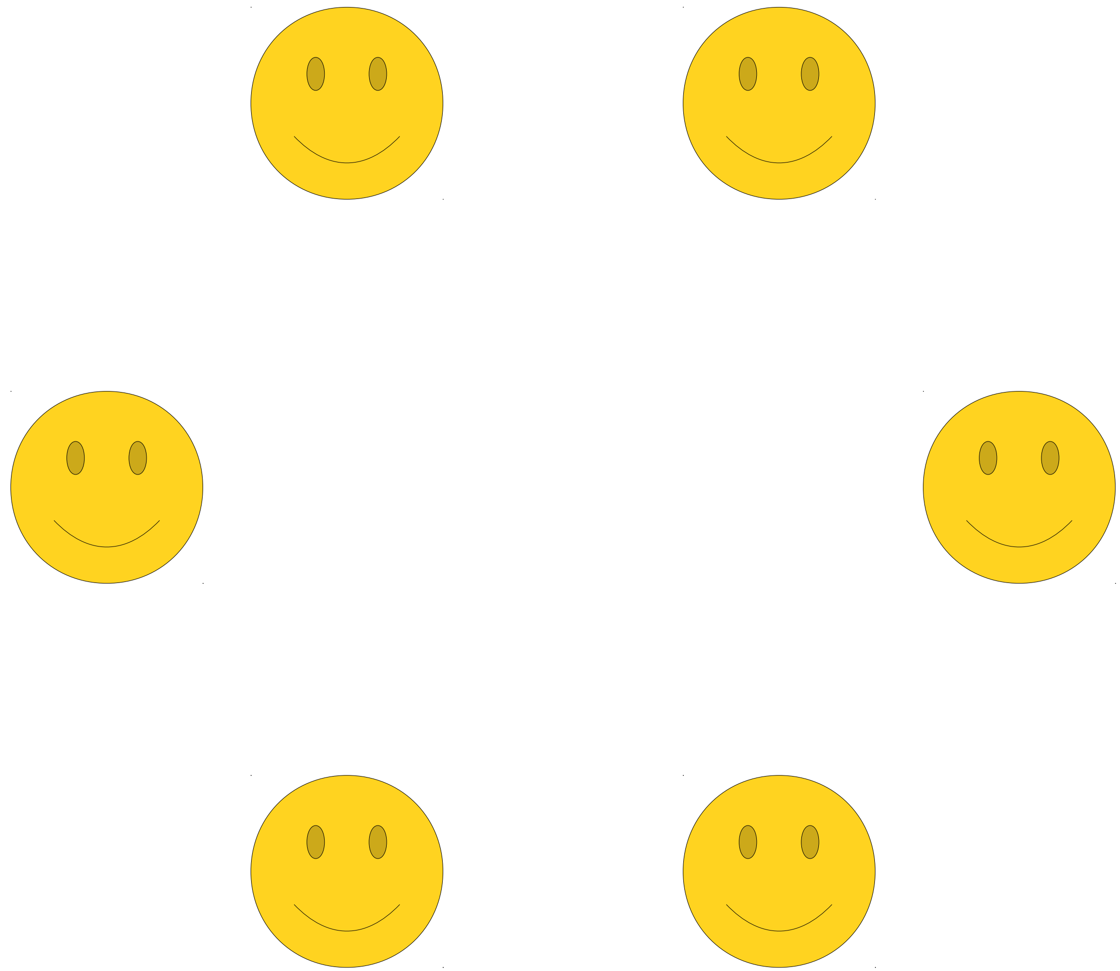
Theorem (The Pigeonhole Principle): If m objects are distributed into n bins and $m > n$, then at least one bin will contain at least two objects.

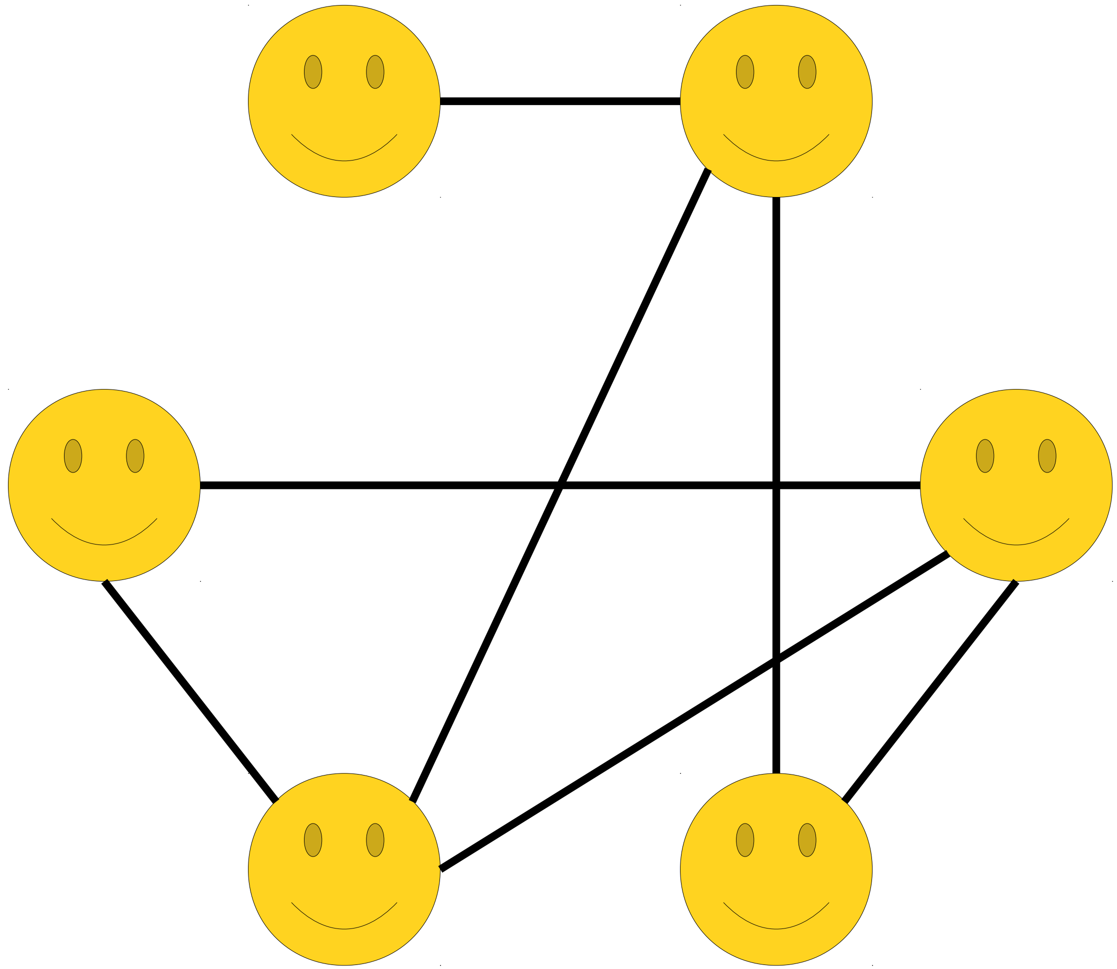
Let A and B be finite sets and assume $|A| > |B|$.
How many of the following statements **must** be true?

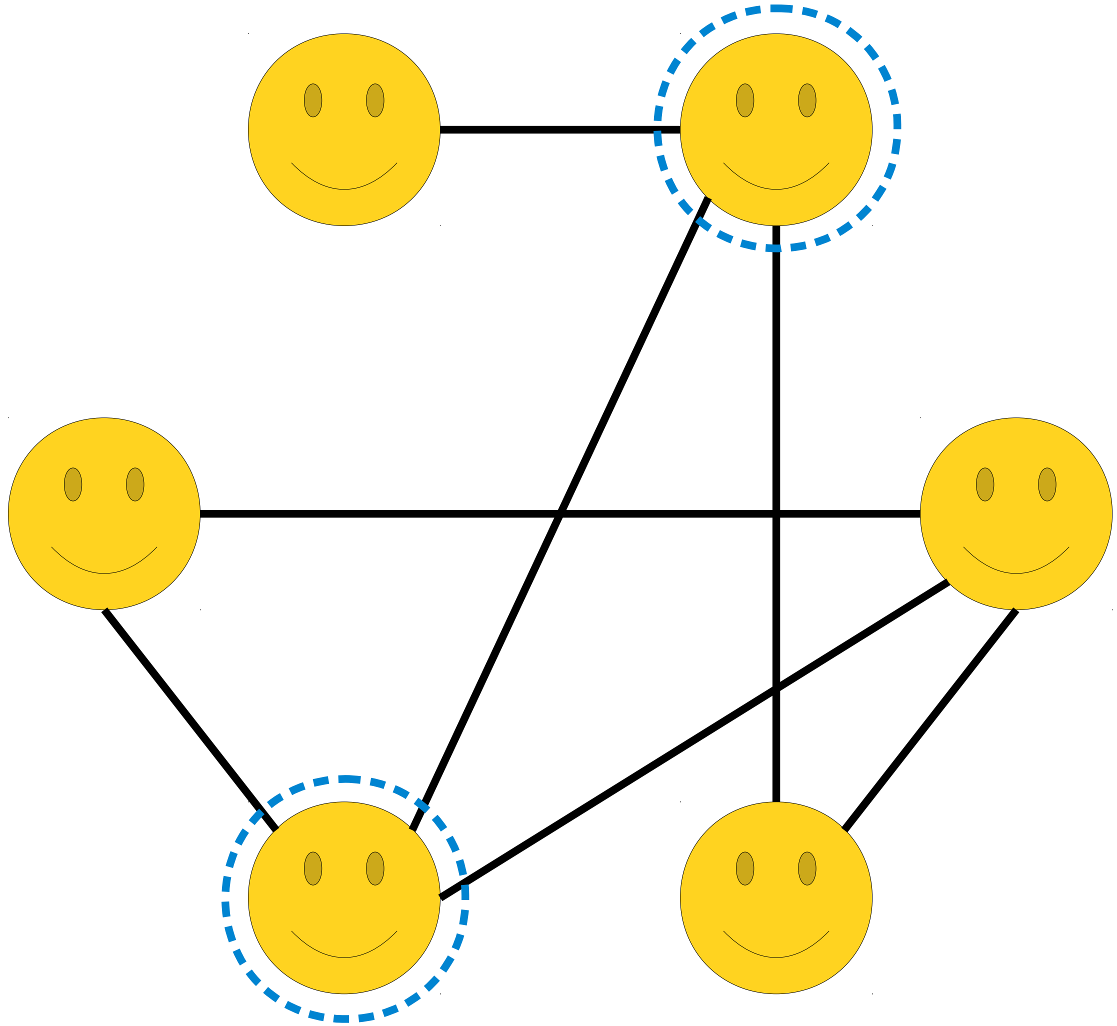
If $f : A \to B$, then f is injective.
If $f : A \to B$, then f is not injective.
If $f : A \to B$, then f is surjective.
If $f : A \to B$, then f is not surjective.

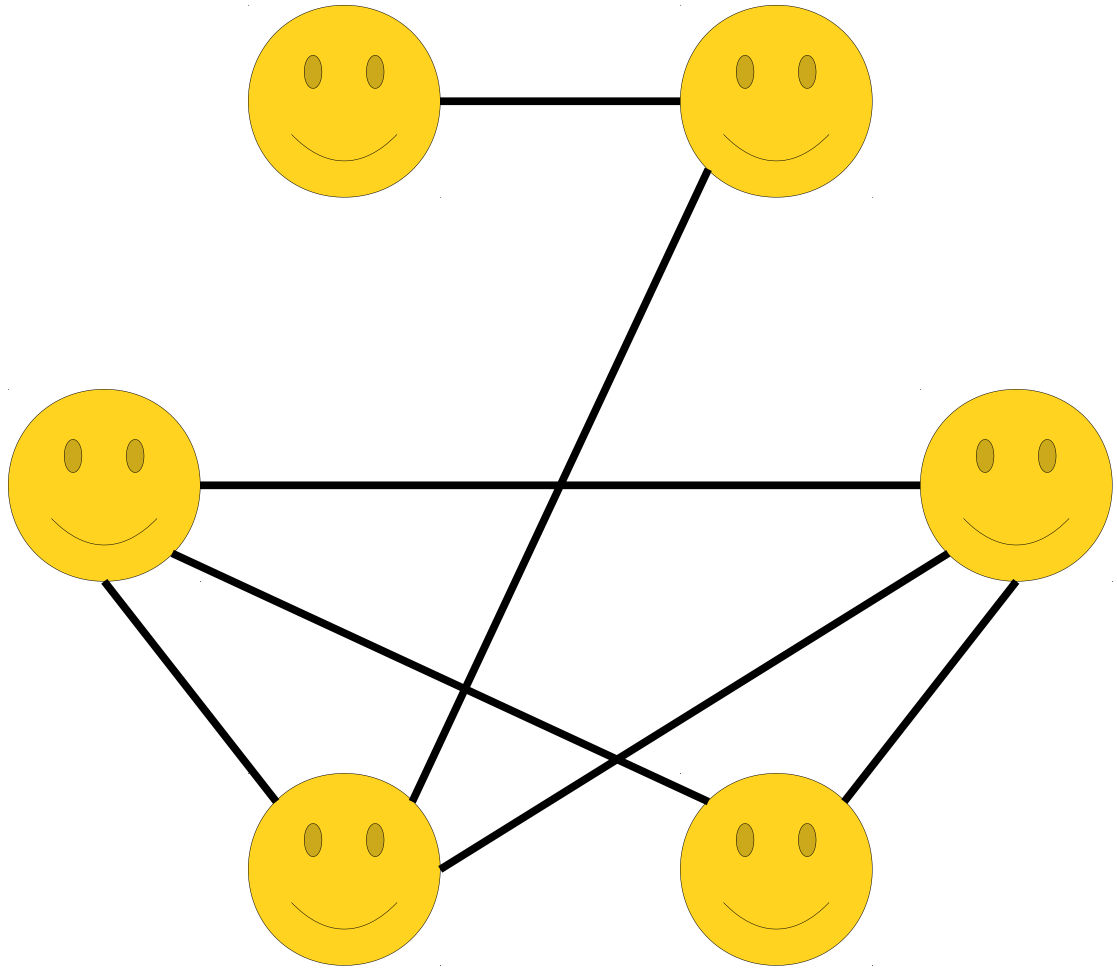
Answer at **PollEv.com/cs103** or
text **CS103** to **22333** once to join, then a number.

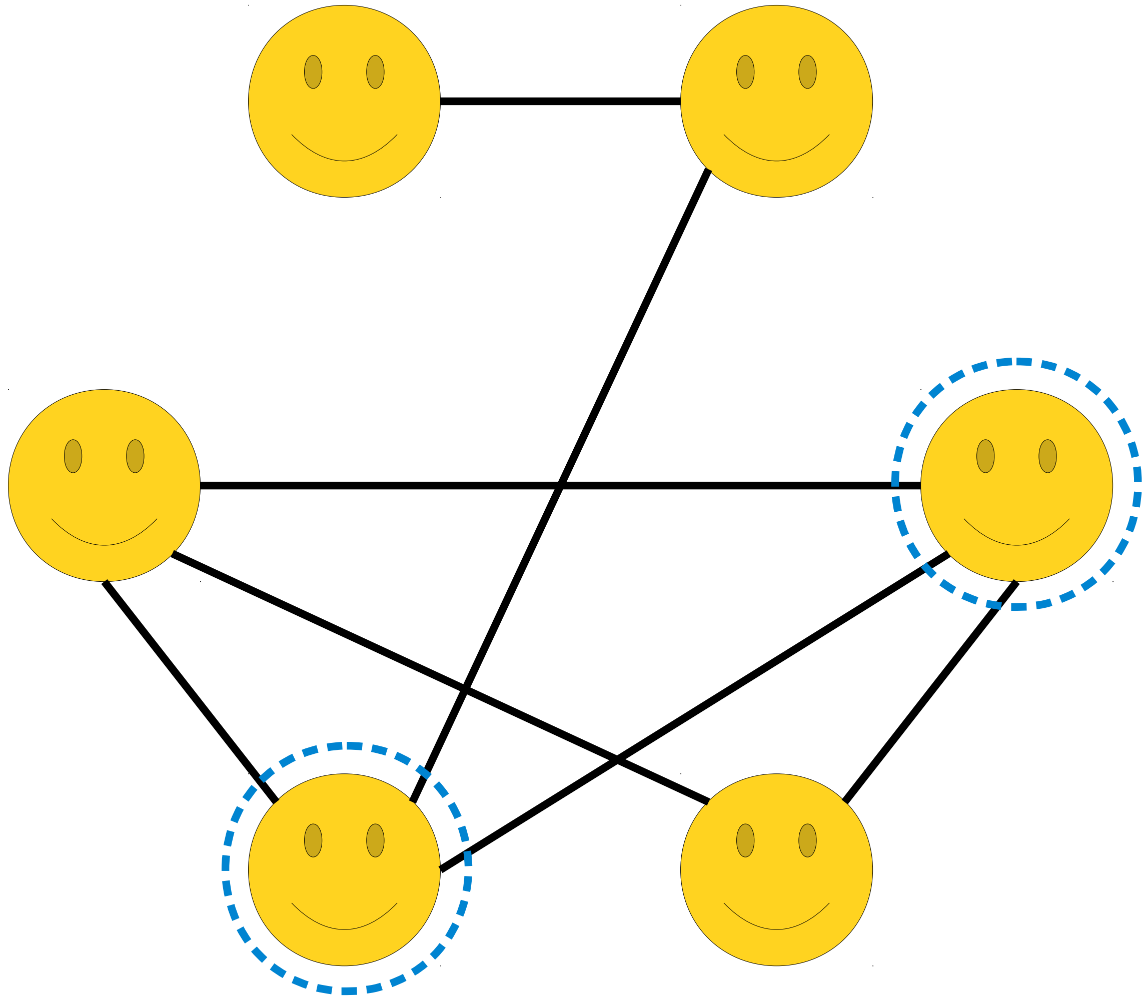
Pigeonhole Principle Party Tricks

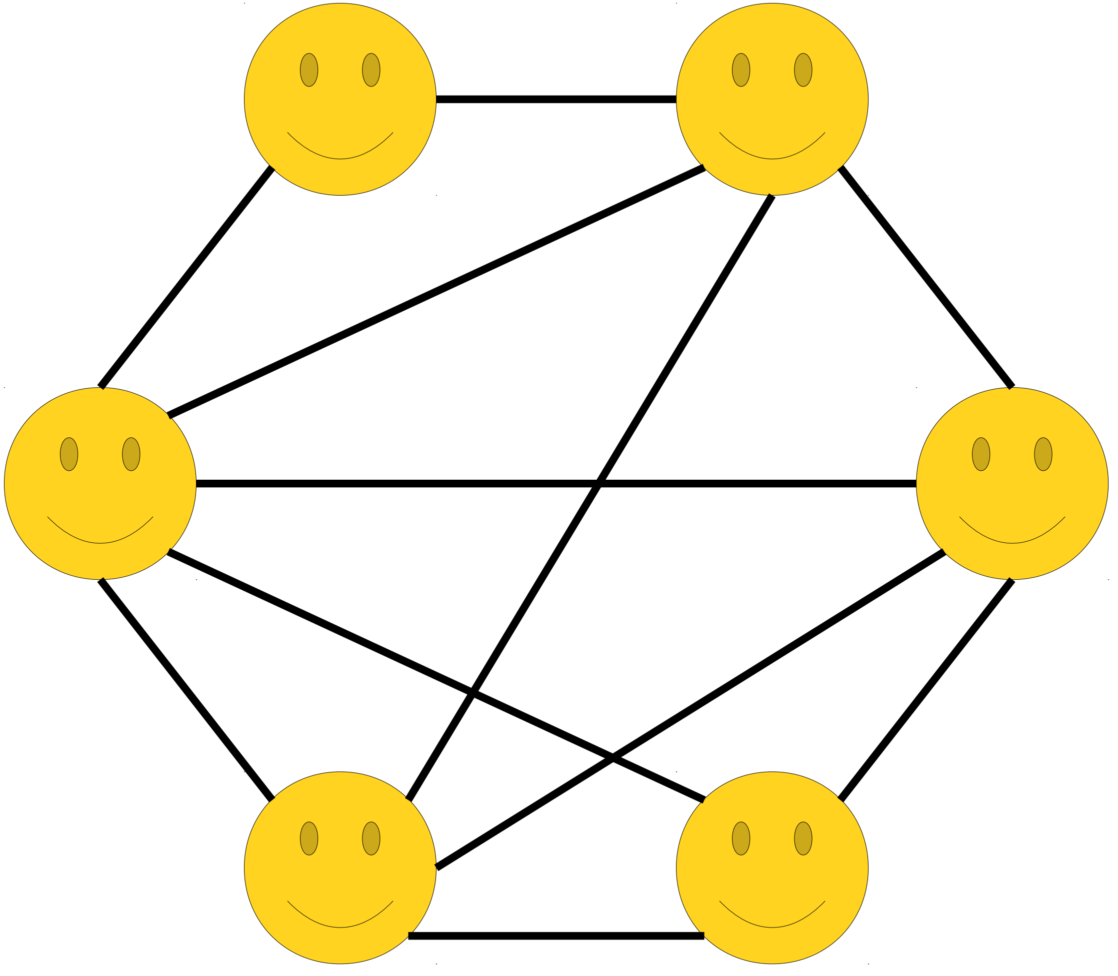


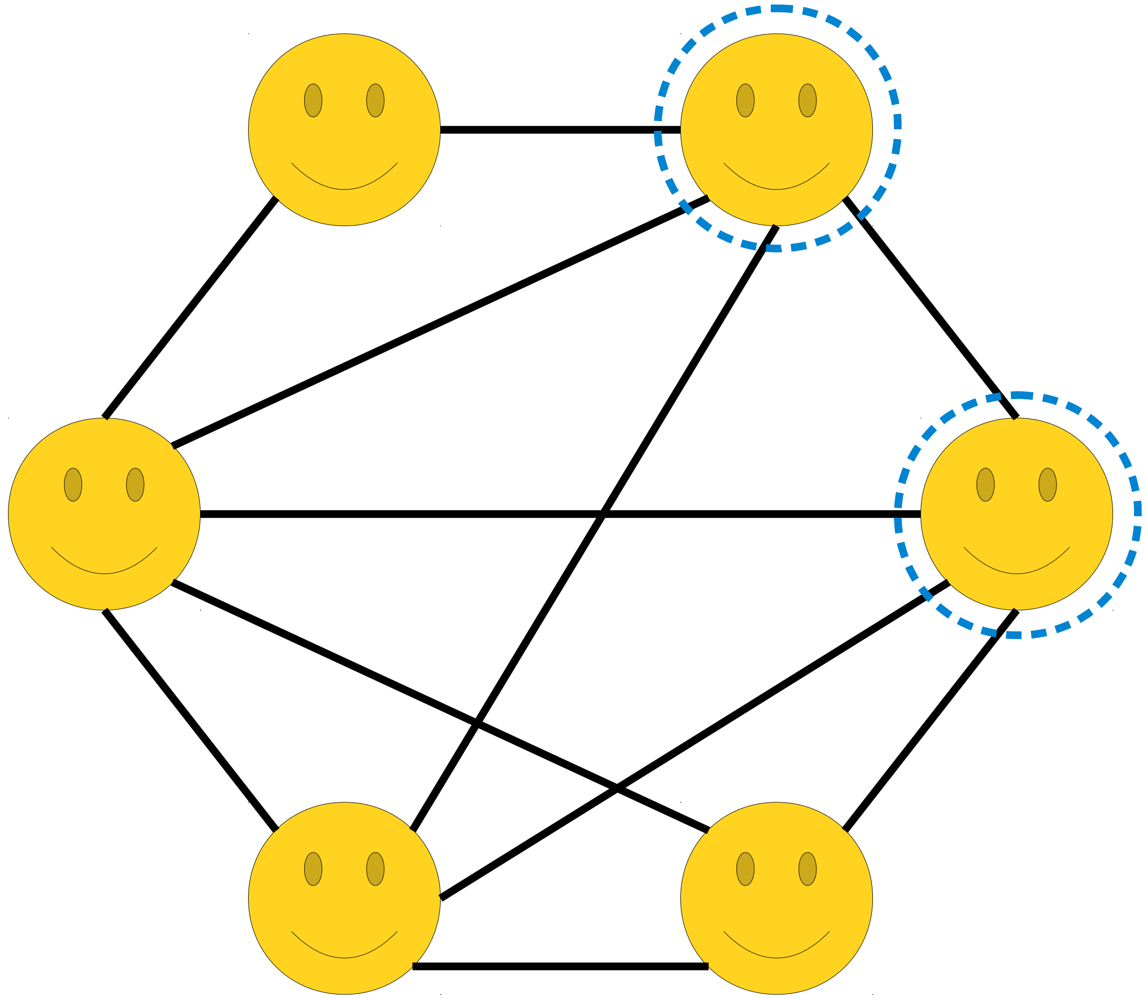






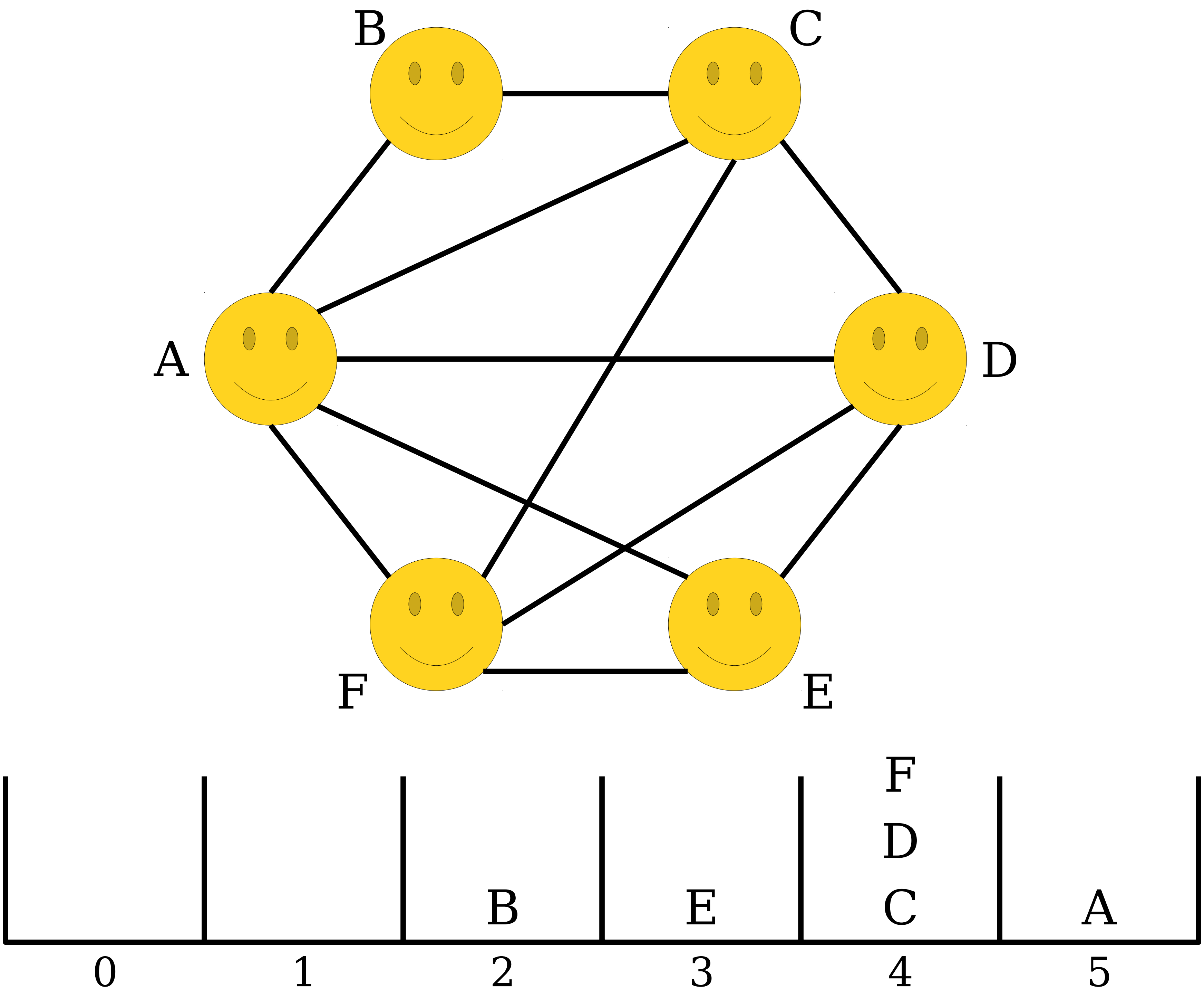






Party Planning

- ***Theorem:*** Every graph with at least two nodes has at least two nodes with the same degree.
  - Equivalently: at any party with at least two people, there are at least two people with the same number of friends at the party.



B
C
A
D
F
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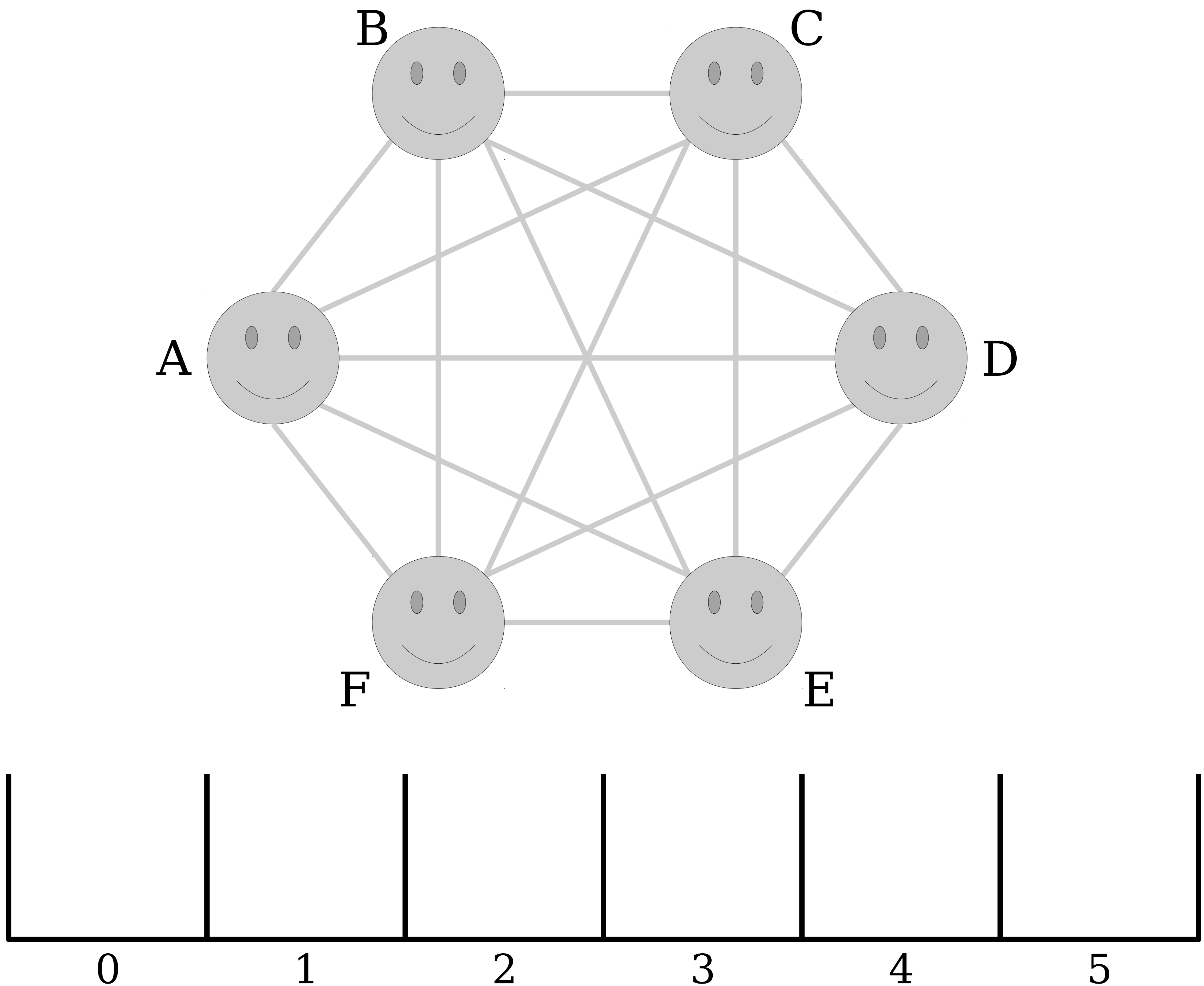
F
D
B
E
C
A
0
1
2
3
4
5


B
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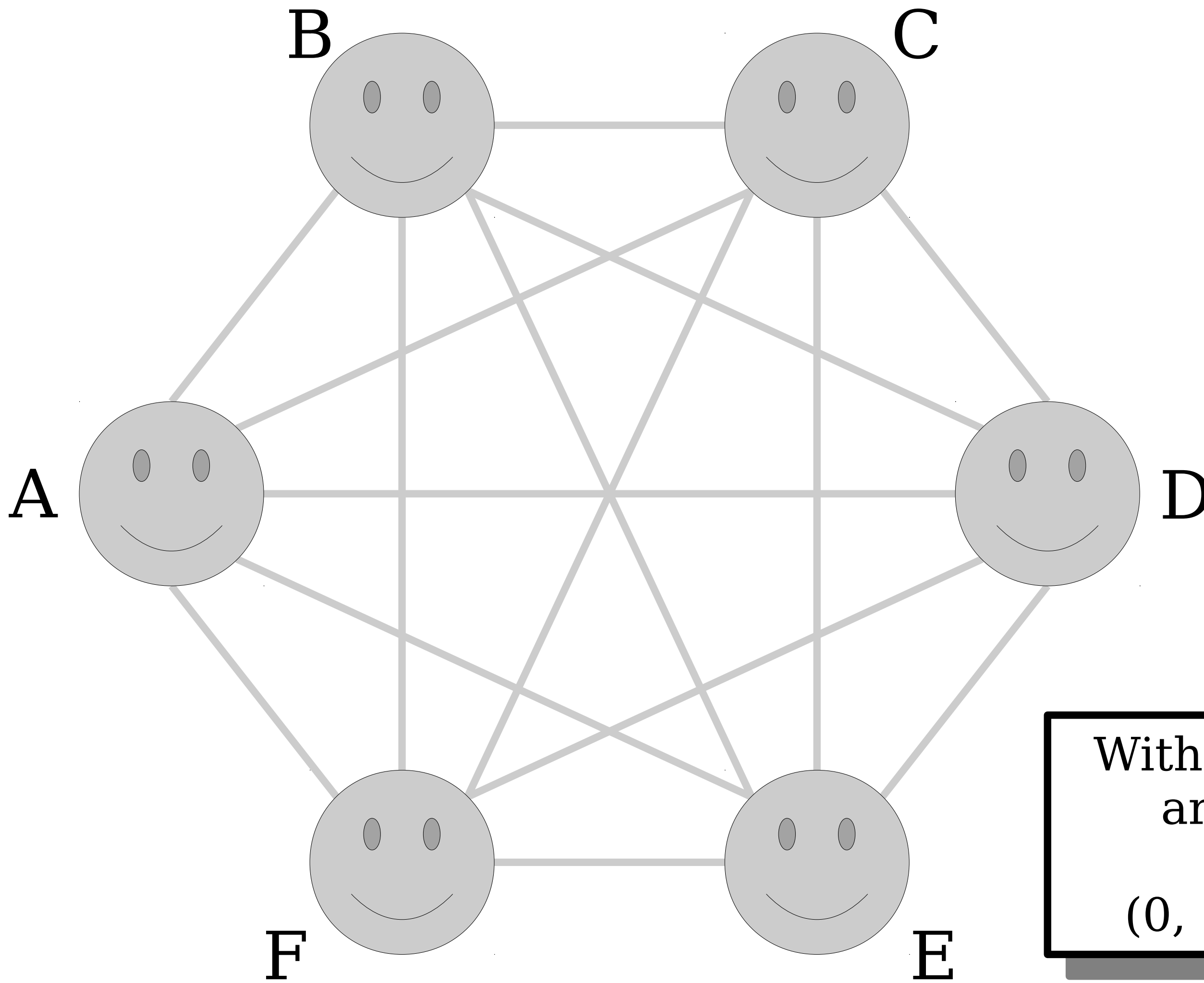
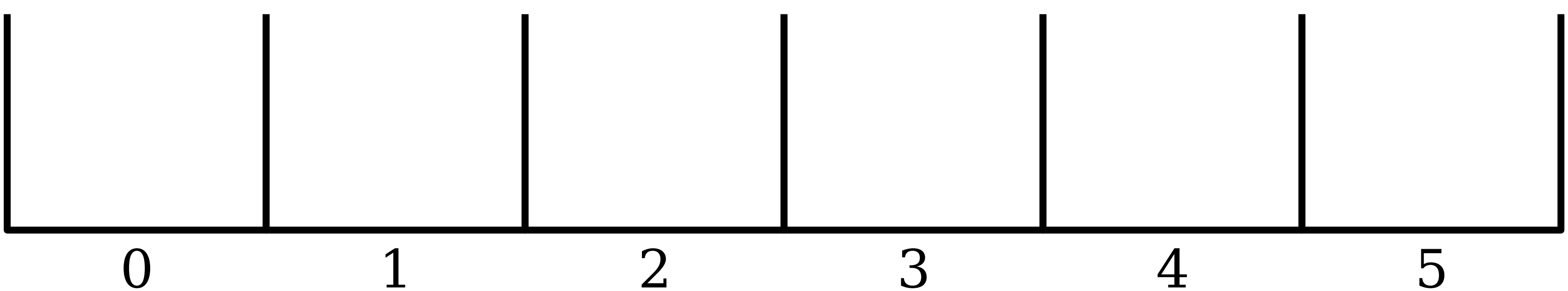
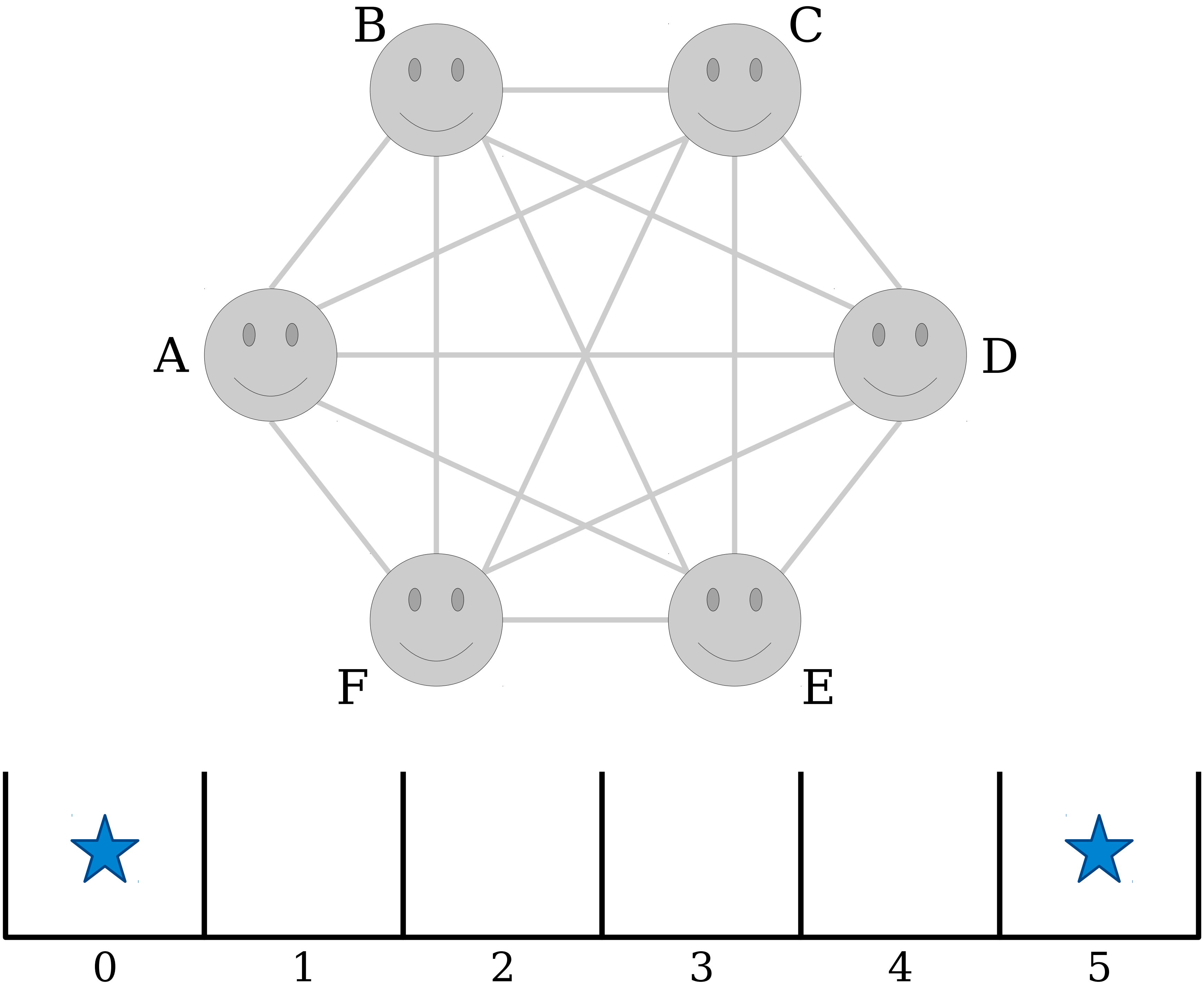


With *n* nodes, there are *n* possible degrees (0, 1, 2, …, *n* – 1)





B
C
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E
0
1
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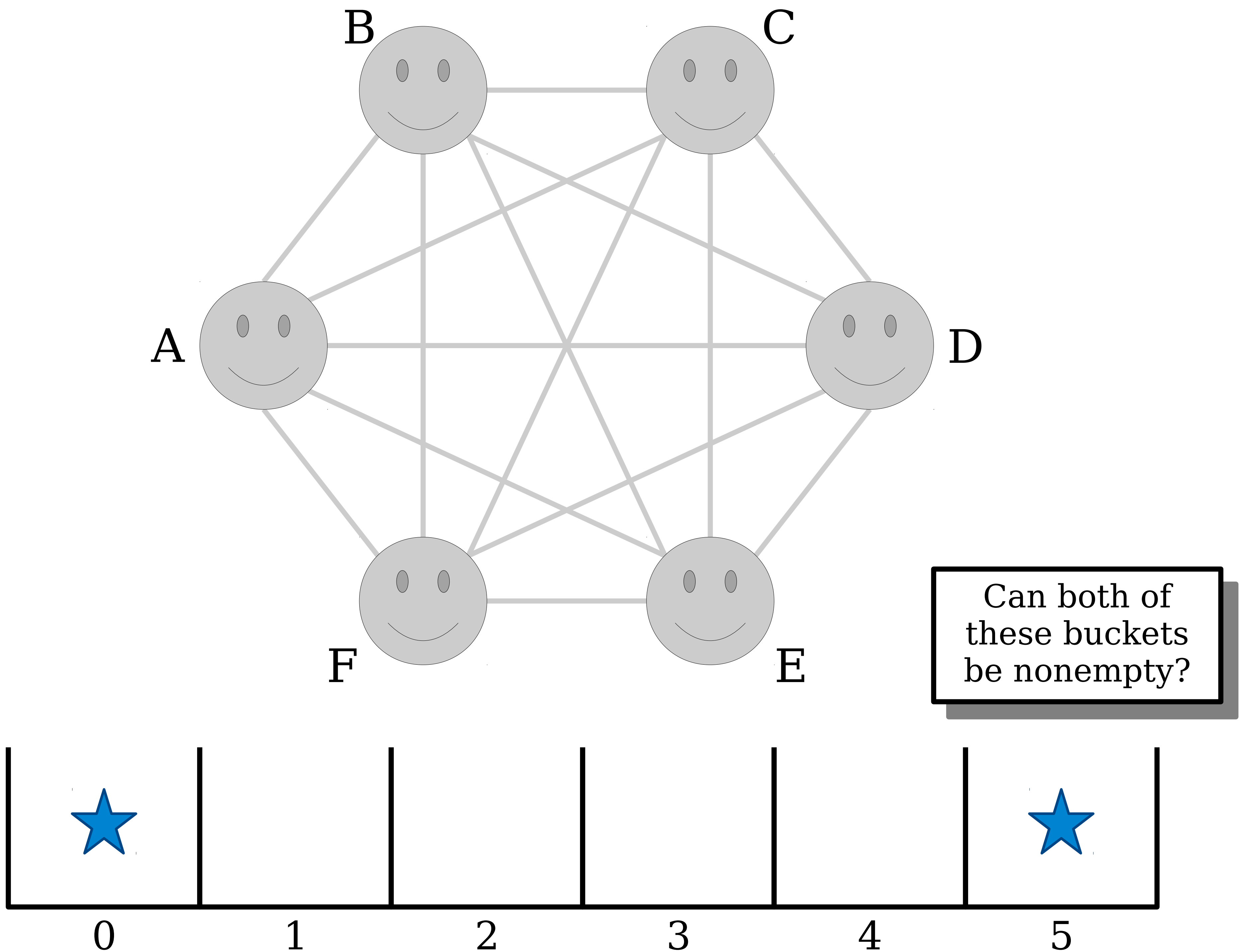
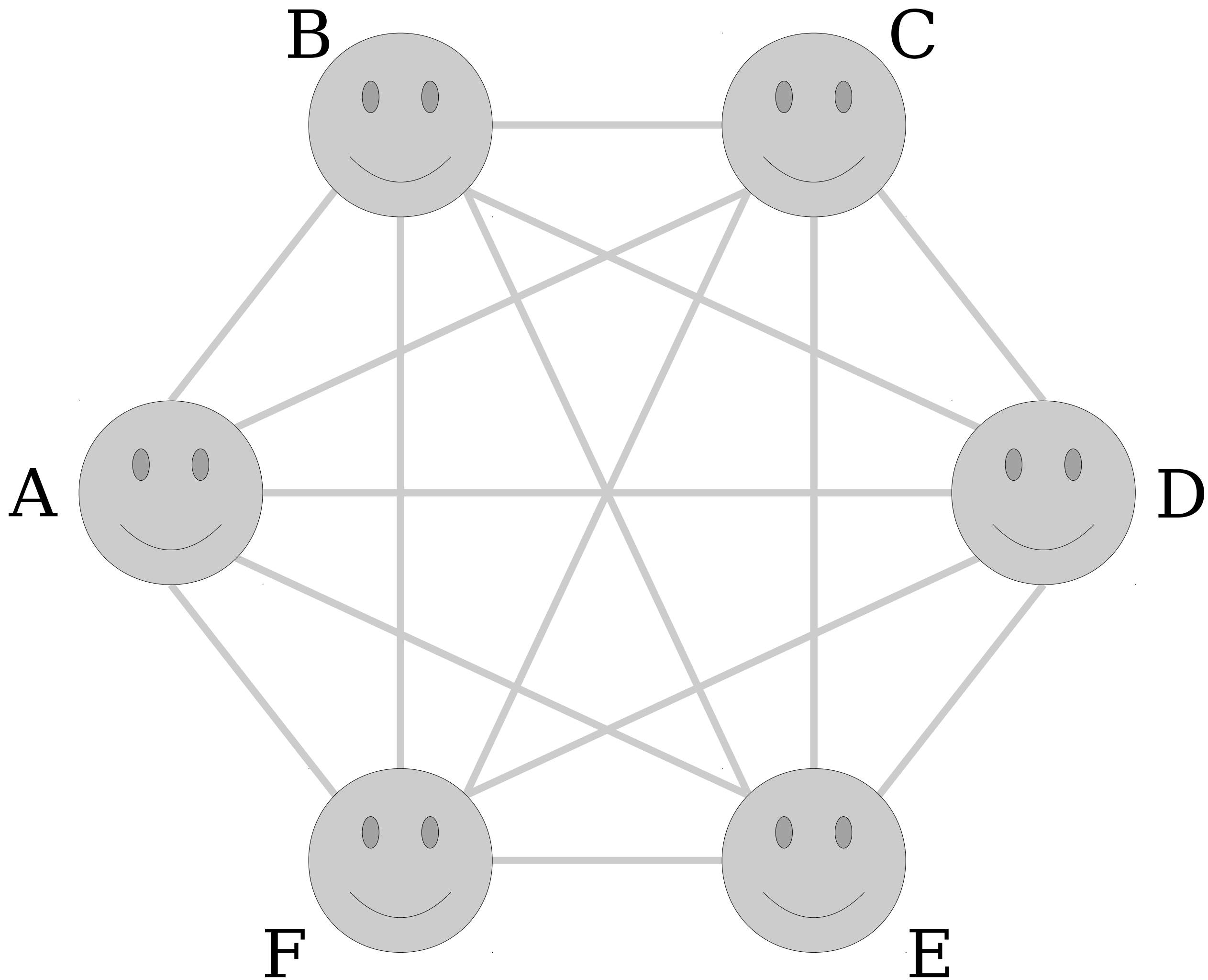
4
5


B
C
A
D
F
E
Can both of these buckets be nonempty?
0
1
2
3
4
5


B
C
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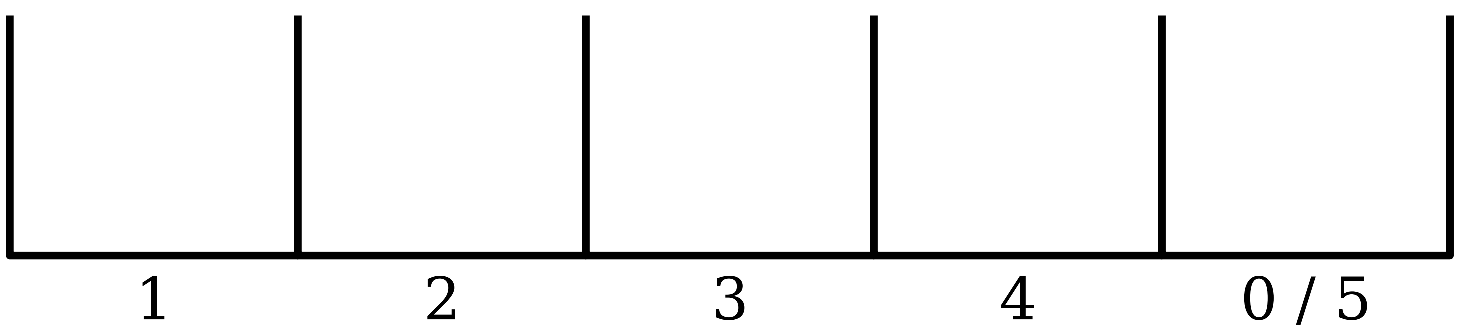
E


1
2
3
4
0 / 5

Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

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We claim that G cannot simultaneously have a node u of degree 0 and a node v of degree $n - 1$:

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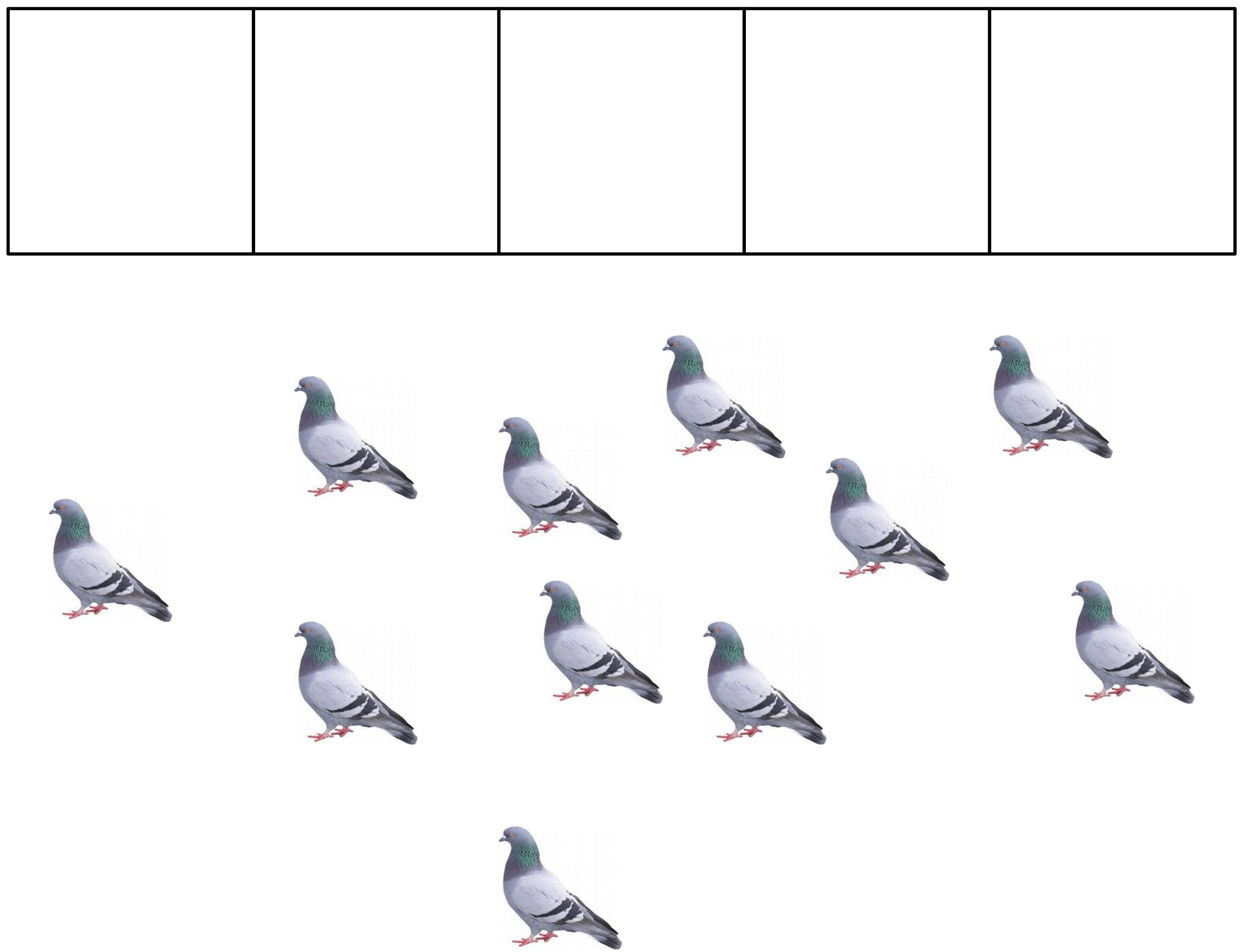
Theorem: In any graph with at least two nodes, there are at least two nodes of the same degree.

Proof 2: Assume for the sake of contradiction that there is a graph G with $n \geq 2$ nodes where no two nodes have the same degree. There are n possible choices for the degrees of nodes in G, namely 0, 1, 2, …, $n - 1$, so this means that G must have exactly one node of each degree. However, this means that G has a node of degree 0 and a node of degree $n - 1$. (These can't be the same node, since $n \geq 2$.) This first node is adjacent to no other nodes, but this second node is adjacent to every other node, which is impossible.

We have reached a contradiction, so our assumption must have been wrong. Thus if G is a graph with at least two nodes, G must have at least two nodes of the same degree. ■

The Generalized Pigeonhole Principle

The Pigeonhole Principle



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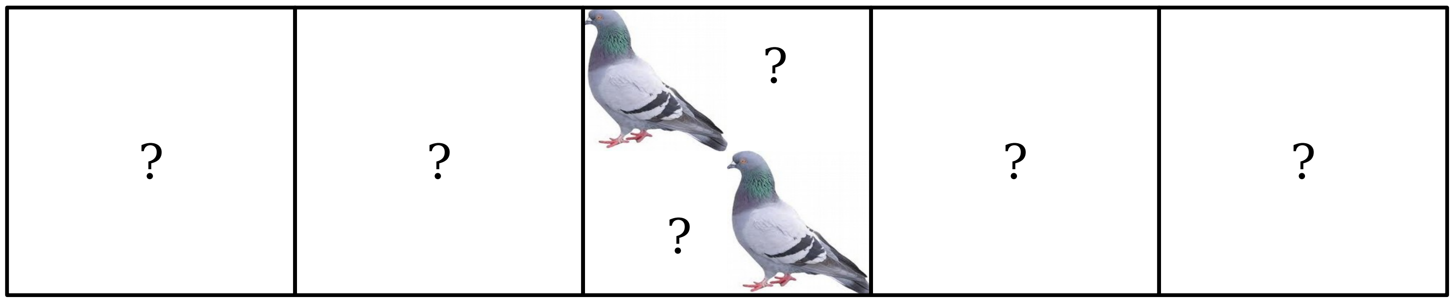

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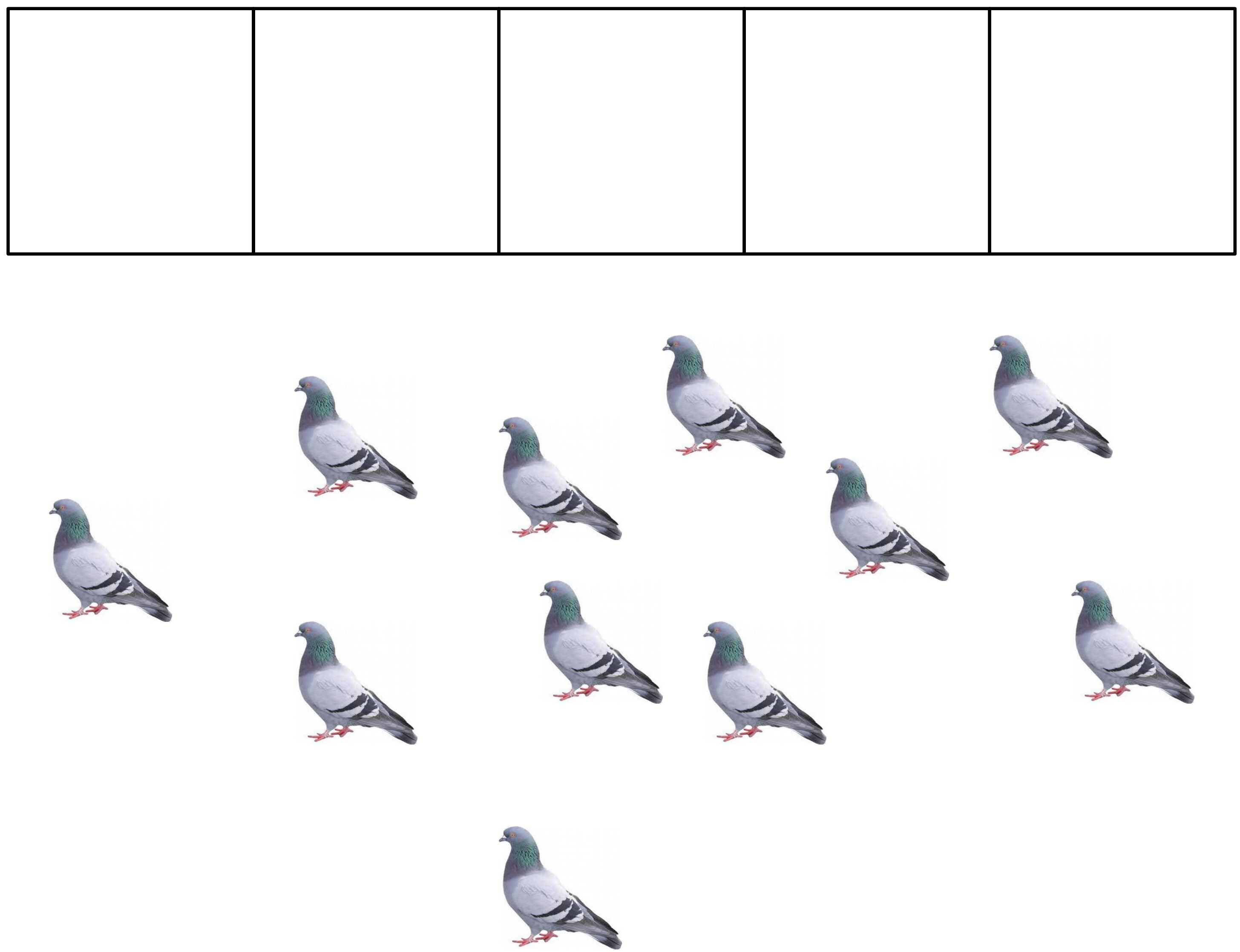

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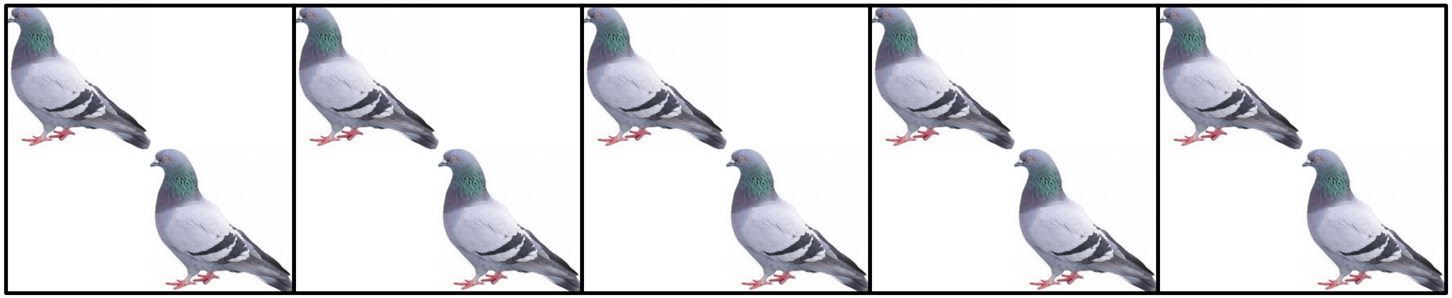

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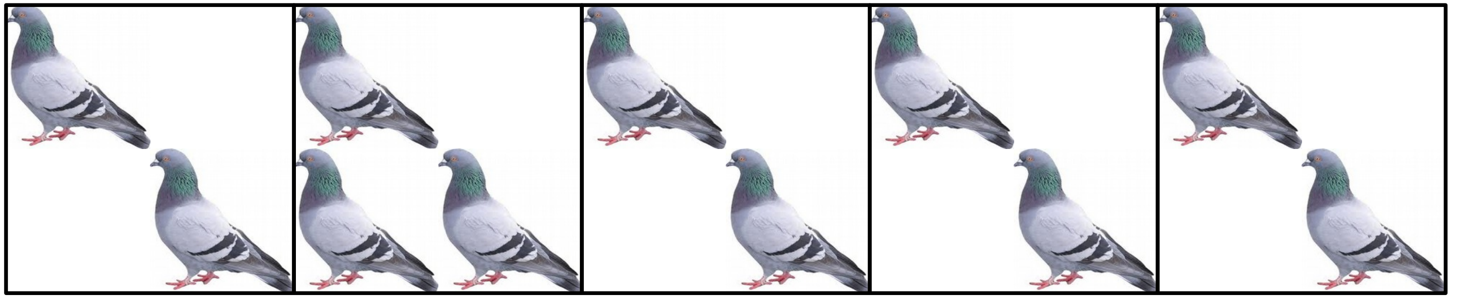

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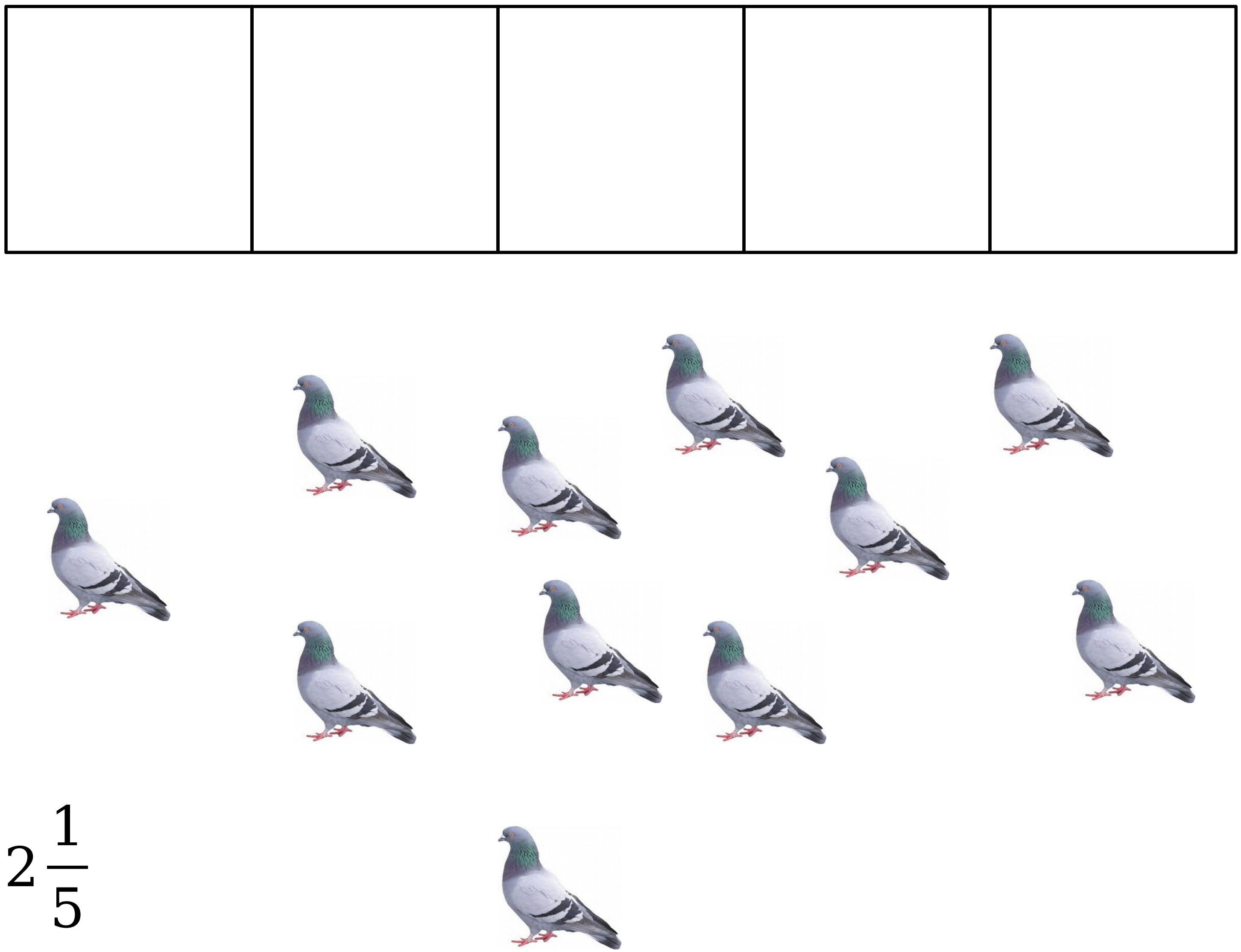
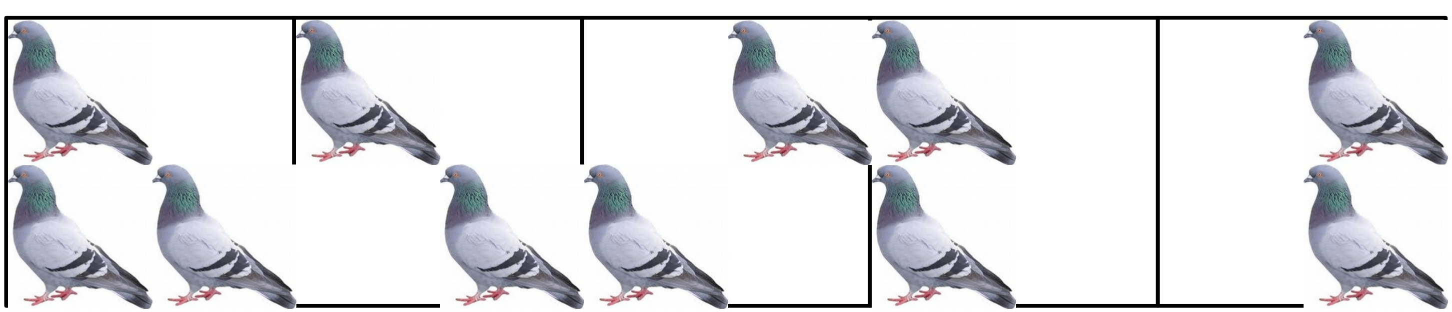


$$\frac{11}{5} = 2\frac{1}{5}$$

A More General Version

- The ***generalized pigeonhole principle*** says that if you distribute *m* objects into *n* bins, then
  - some bin will have at least $\lceil {}^{m}/_{n} \rceil$ objects in it, and
  - some bin will have at most $\lfloor {}^{m}/_{n} \rfloor$ objects in it.

$\lceil {}^{m}/_{n} \rceil$ means "${}^{m}/_{n}$, rounded up."
$\lfloor {}^{m}/_{n} \rfloor$ means "${}^{m}/_{n}$, rounded down."



$m = 11$
$n = 5$

$\lceil m / n \rceil = 3$
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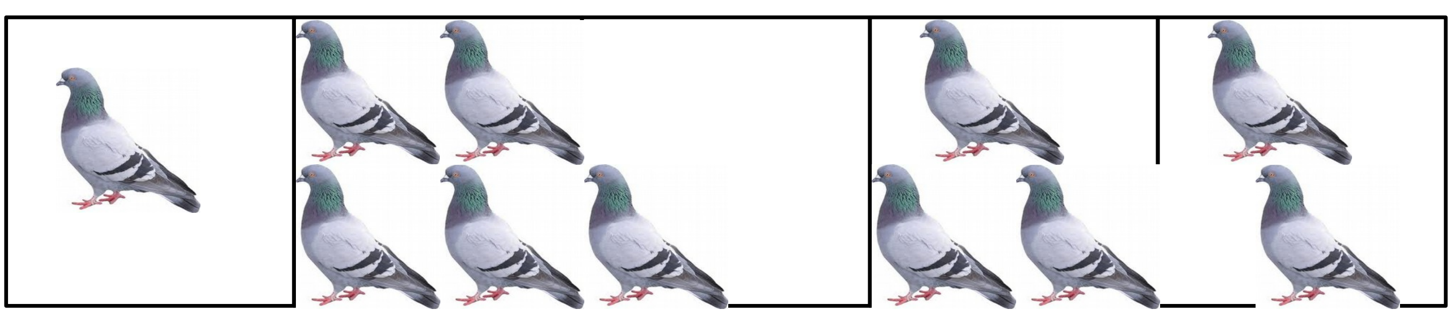

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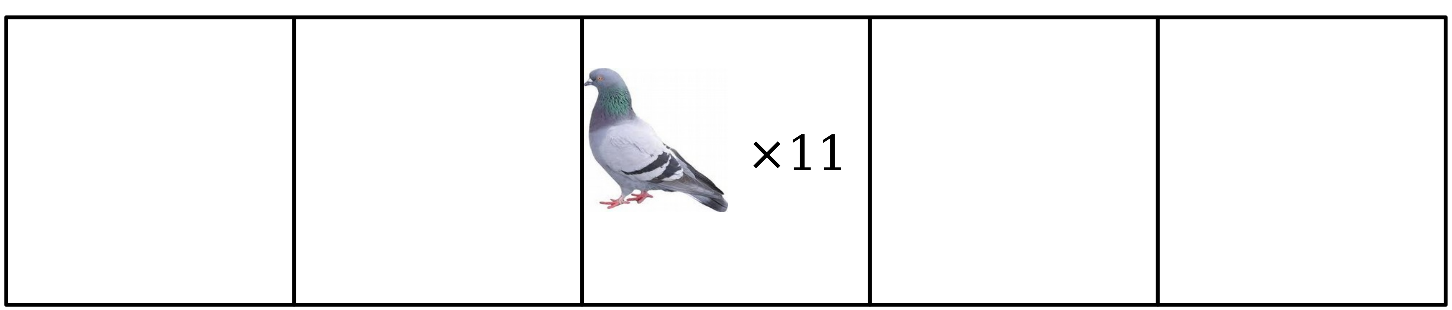

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An Application: Friends and Strangers

Friends and Strangers

- Suppose you have a party of six people. Each pair of people are either friends (they know each other) or strangers (they do not).
- ***Theorem:*** Any such party must have a group of three mutual friends (three people who all know one another) or three mutual strangers (three people, none of whom know any of the others).

Friends and Strangers Restated

- From a graph-theoretic perspective, the Theorem on Friends and Strangers can be restated as follows:

  Theorem: Any 6-clique whose edges are colored red and blue contains a red triangle or a blue triangle (or both).

- How can we prove this?

Theorem: Consider a 6-clique in which every edge is colored either red or blue. Then there must be a triangle of red edges, a triangle of blue edges, or both.

Proof: Color the edges of the 6-clique either red or blue arbitrarily. Let x be any node in the 6-clique. It is incident to five edges and there are two possible colors for those edges. Therefore, by the generalized pigeonhole principle, at least $\lceil 5/2 \rceil = 3$ of those edges must be the same color. Call that color c_1 and let the other color be c_2.

Let r, s, and t be three of the nodes adjacent to node x along an edge of color c_1. If any of the edges $\{r, s\}$, $\{r, t\}$, or $\{s, t\}$ are of color c_1, then one of those edges plus the two edges connecting back to node x form a triangle of color c_1. Otherwise, all three of those edges are of color c_2, and they form a triangle of color c_2. Overall, this gives a red triangle or a blue triangle, as required. ■

Ramsey Theory

- The proof we did is a special case of a broader result.
- ***Theorem (Ramsey's Theorem):*** For any natural number n, there is a smallest natural number $R(n)$ such that if the edges of an $R(n)$-clique are colored red or blue, the resulting graph will contain either a red n-clique or a blue n-clique.
  - Our proof was that $R(3) \leq 6$.